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2009 J. Phys. A: Math. Theor. 42 265304

(http://iopscience.iop.org/1751-8121/42/26/265304)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 265304 (13pp) doi:10.1088/1751-8113/42/26/265304

# Quasiperiodic surface Maryland models on quantum graphs

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Received 9 December 2008, in final form 13 May 2009
Published 9 June 2009
Online at stacks.iop.org/JPhysA/42/265304

**Abstract**
We study quantum graphs corresponding to isotropic lattices with quasiperiodic coupling constants given by the same expressions as the coefficients of the discrete surface Maryland model. The absolutely continuous and the pure point spectra are described. It is shown that the transition between them is governed by the Hill operator corresponding to the edge potential.

PACS numbers: 02.30.Tb, 71.23.−k
Mathematics Subject Classification: 81Q10, 47B39, 47N50, 82B44

## 1. Introduction

The present paper is devoted to the spectral analysis of a special class of quasiperiodic interactions on quantum graphs. We are going to show how some topics of the theory of discrete quasiperiodic operators can be transferred to the quantum graph case using the operator-theoretic tools.

Let us introduce first a class of discrete quasiperiodic potentials. Take a positive integer $d$, pick $d_1 \in \{0, \ldots, d\}$ and set $d_2 := d - d_1$. In what follows we represent any $\mathbf{m} \in \mathbb{Z}^d$ as $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2)$ with $\mathbf{m}_1 \in \mathbb{Z}^{d_1}$ and $\mathbf{m}_2 \in \mathbb{Z}^{d_2}$. Pick now $g \neq 0$, $\boldsymbol{\omega} \in \mathbb{R}^{d_2}$, $\varphi \in \mathbb{R}$ with

$$\varphi \neq \boldsymbol{\omega}\mathbf{m}_2 \mod \tfrac{1}{2}, \qquad \mathbf{m}_2 \in \mathbb{Z}^{d_2} \tag{1}$$

and set

$$\alpha(\mathbf{m}) := g \tan \pi(\boldsymbol{\omega}\mathbf{m}_2 + \varphi), \qquad m \in \mathbb{Z}^d. \tag{2}$$

Consider also the discrete Laplacian $\Delta$ in $\ell^2(\mathbb{Z}^d)$ given by

$$\Delta_d f(\mathbf{m}) = \sum_{\mathbf{m}':|\mathbf{m}-\mathbf{m}'|=1} f(\mathbf{m}').$$

The operators of the form $H = \Delta_d + \alpha$ are usually referred to as Maryland-type models.

The paper [12] dealt with the case $d = d_2 = 1$ which provided the first explicit example of a difference quasiperiodic operator having a dense pure point spectrum everywhere; this operator is often referred to as the classical Maryland model. Later the class of such Hamiltonians was considerably extended in several directions, e.g. to the multidimensional case and to more general unperturbed operators, see e.g. [1, 9].

The papers [2, 3, 13] studied the situation $0 < d_1 < d$; in this case the potential is supported by a subspace, and the corresponding operator is referred to as the surface Maryland model. The quasiperiodic perturbation leaves unchanged the absolutely continuous spectrum of the unperturbed operator (i.e. of the Laplacian) but produces (under some incommensurability conditions) a dense pure point spectrum on the rest of the real line.

On the other hand, discrete operators are closely related to the quantum graph models, i.e. differential operators acting on geometric configurations consisting of segments, see e.g. [6, 11, 14, 15]. The aim of the present paper is to provide an analog of the surface Maryland model for quantum graphs and to study its spectral properties. The work is a natural continuation of our previous paper [18] where we considered the full-space Maryland quantum graph model ($d_1 = 0$).

We are studying an isotopic quantum graph lattice, and the above coefficients $\alpha(\mathbf{m})$ determine the strengths of the $\delta$-potentials placed at the corresponding vertices. We note that coupling constants at the vertices may be used to approximate rather general Schrödinger operators, see e.g. [8, 16]. The spectral problem for the quantum graph can then be reduced to a nonlinear spectral problem for an energy-dependent surface Maryland model using the theory of self-adjoint extensions [5]. The reduced operator is then studied using a combination of the machinery of Weyl functions [5] and the constructions of [2, 9] for the corresponding discrete operators. We describe the regions of the pure point and the absolutely continuous spectra and show that their location is controlled by the one-dimensional Hill operator associated with the potential on the edges. We believe that the problem is also of interest for the theory of self-adjoint extensions, as it provides an explicit example of a transition between different spectral types described using the Weyl functions, which is an extremely difficult problem in the general setting [4].

## 2. Model operator and main results

As mentioned in section 1, we are studying the simple quantum graph lattices. The vertex set is $\mathbb{Z}^d$, and the edges are between nearest neighbors. The length of each edge is 1, and each edge carries the differential operator $-\mathrm{d}^2/\mathrm{d}x^2 + q$ with the same real $L^2$-potential $q$, and the boundary conditions at the vertices are of the $\delta$-type with the coupling constants given by (2). Let us introduce a notation to handle this situation in detail.

The set of graph vertices is $\mathbb{Z}^d$, $d \geqslant 2$ (i.e. we explicitly need a multidimensional lattice). By $\mathbf{h}_j$, $j = 1, \dots, d$, we denote the standard basis vectors of $\mathbb{Z}^d$. For technical reasons we need an orientation on each edge. Two vertices $\mathbf{m}, \mathbf{m}'$ are connected by an oriented edge $\mathbf{m} \to \mathbf{m}'$ iff $\mathbf{m}' = \mathbf{m} + \mathbf{h}_j$ for some $j \in \{1, \dots, d\}$; this edge is denoted as $(\mathbf{m}, j)$ and one says that $\mathbf{m}$ is its initial vertex and $\mathbf{m}' \equiv \mathbf{m} + \mathbf{h}_j$ is its terminal vertex.

Replace now each edge $(\mathbf{m}, j)$ by a copy of the segment $[0, 1]$ in such a way that 0 is identified with $\mathbf{m}$ and 1 is identified with $\mathbf{m} + \mathbf{h}_j$. In this way we arrive at a certain topological set carrying a natural metric structure. The quantum state space of the system is

$$\mathcal{H} := \bigoplus_{(\mathbf{m},j)\in\mathbb{Z}^d\times\{1,\dots,d\}} \mathcal{H}_{\mathbf{m},j}, \qquad \mathcal{H}_{\mathbf{m},j} = \mathcal{L}^2[0,1],$$

and the vectors $f \in \mathcal{H}$ will be denoted as $f = (f_{\mathbf{m},j})$, $f_{\mathbf{m},j} \in \mathcal{H}_{\mathbf{m},j}$, $\mathbf{m} \in \mathbb{Z}^d$, $j = 1, \dots, d$.

To introduce a Schrödinger operator acting in $\mathcal{H}$ let us fix a real-valued potential $q \in \mathcal{L}^2[0,1]$ and some real constants $\alpha(\mathbf{m}), \mathbf{m} \in \mathbb{Z}^d$. Set $A := \operatorname{diag}(\alpha(\mathbf{m}))$; this is a self-adjoint operator in $\ell^2(\mathbb{Z}^d)$. Denote by $H_A$ the operator acting as

$$(f_{\mathbf{m},j}) \mapsto (-f''_{\mathbf{m},j} + qf_{\mathbf{m},j}), \tag{3a}$$

on functions $f = (f_{\mathbf{m},j}) \in \bigoplus_{\mathbf{m},j} H^2[0,1]$ satisfying the following boundary conditions:

$$f_{\mathbf{m},j}(0) = f_{\mathbf{m}-\mathbf{h}_k,k}(1) =: f(\mathbf{m}), \qquad j,k = 1,\dots,d, \qquad \mathbf{m} \in \mathbb{Z}^d \tag{3b}$$

(which means the continuity at all vertices) and

$$f'(\mathbf{m}) = \alpha(\mathbf{m}) f(\mathbf{m}), \qquad \mathbf{m} \in \mathbb{Z}^d, \tag{3c}$$

where

$$f'(\mathbf{m}) := \sum_{j=1}^{d} f'_{\mathbf{m},j}(0) - \sum_{j=1}^{d} f'_{\mathbf{m}-\mathbf{h}_j,j}(1). \tag{3d}$$

The constants $\alpha(\mathbf{m})$ are usually referred to as *Kirchhoff coupling constants* and interpreted as the strengths of zero-range impurity potentials placed at the corresponding vertices, cf [6]. The zero coupling constants correspond hence to the ideal couplings.

We are going to study the above operator $H_A$ for the coupling constants $\alpha(\mathbf{m})$ given by (2) where $1 \leqslant d_1 < d$. This operator will be noted simply by $H$.

To formulate the results we need some additional constructions. Denote by $s$ and $c$ the solutions to $-y'' + qy = zy$ satisfying $s(0;z) = c'(0;z) = 0$ and $s'(0;z) = c(0;z) = 1$, $z \in \mathbb{C}$, and set $\eta(z) := s(1;z) + c'(1;z)$. Consider an auxiliary one-dimensional Hill operator

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + Q, \qquad Q(x+n) = q(x), \qquad (x,n) \in [0,1) \times \mathbb{Z}. \tag{4}$$

It is known that $\operatorname{spec} L = \eta^{-1}([-2,2])$ and the spectrum is absolutely continuous and has a band structure, i.e. is a locally finite union of segments.

The following theorem summarizes propositions 5, 6, 10 and 13 below and contains the main results.

**Theorem 1.** *For any $\boldsymbol{\omega}$ and $\varphi$ one has* $\operatorname{spec} L \subset \operatorname{spec} H$. *If the components of $\boldsymbol{\omega}$ are rationally independent, then the spectrum of $H$ in $\eta^{-1}((-2,2))$ is purely absolutely continuous. If $\boldsymbol{\omega}$ satisfies additionally the Diophantine condition*

$$\text{there are } C, \beta > 0 \quad \text{with} \quad |\boldsymbol{\omega}\mathbf{m}_2 - r| \geqslant C|\mathbf{m}_2|^{-\beta} \quad \text{for all} \quad \mathbf{m}_2 \in \mathbb{Z}^{d_2}\setminus\{\mathbf{0}\}, \quad r \in \mathbb{Z}, \tag{5}$$

*then the spectrum of H covers the whole real line and is pure point outside* $\operatorname{spec} L$.

As easily seen, the absolutely continuous spectrum of $H$ just coincides with the spectrum of $L$ (which is independent of the quasiperiodic perturbation), and, under the additional assumptions, the rest of the spectrum is pure point. It is interesting to mention that a similar interlaced spectrum was found recently in a completely different model involving singular potentials [7].

## 3. Resolvents for quantum graphs

Denote by $S$ the operator acting as (3a) on the functions $f$ satisfying only the boundary conditions (3b). On the domain of $S$ one can define linear maps

$$f \mapsto \Gamma f := (f(\mathbf{m}))_{\mathbf{m}\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d), \qquad f \mapsto \Gamma' f := (f'(\mathbf{m}))_{\mathbf{m}\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d).$$

By direct calculation one can see that for any $f, g \in \operatorname{dom} S$ there holds $\langle f, Sg\rangle - \langle Sf, g\rangle = \langle \Gamma f, \Gamma' g\rangle - \langle \Gamma' f, \Gamma g\rangle$ and that for any $\xi, \xi' \in \mathbb{Z}^d$ there exists $f \in \operatorname{dom} S$ with $\Gamma f = \xi$ and $\Gamma' f = \xi'$. From the operator-theoretic point of view $(\mathbb{Z}^d, \Gamma, \Gamma')$ is a boundary triple for $S$ and hence a powerful machinery based on the Krein resolvent formula applies [5].

Note that the restriction of $S$ to the vectors $f$ satisfying $\Gamma f = 0$ is just the direct sum of the operators $-\mathrm{d}^2/\mathrm{d}x^2 + q$ with the Dirichlet boundary conditions over all the segments. The spectrum of $H^0$ is a discrete set and will be referred to as the Dirichlet spectrum of the graph. In its turn, the operator $H$ is the restriction of $S$ to the vectors $f$ satisfying $\Gamma' f = A\Gamma f$ where $A = \operatorname{diag}(\alpha(\mathbf{m}))$.

Let $z \notin \operatorname{spec} H^0$. For $g \in \mathcal{G}$ denote by $\gamma(z)g$ the unique solution to the abstract boundary value problem $(S - z)f = 0$ with $\Gamma f = g$. In our case one can easily calculate

$$(\gamma(z)\xi)_{\mathbf{m},j}(t) = \frac{1}{s(1;z)}(\xi(\mathbf{m}+\mathbf{h}_j)s(t;z) + \xi(\mathbf{m})(s(1;z)c(t;z) \\ - c(1;z)s(t;z))), \qquad t \in [0,1], \qquad (\mathbf{m}, j) \in \mathbb{Z}^d \times \{1,\dots,d\}.$$

The map $\gamma(z)$ is called the $\gamma$-field associated with the boundary triple. The corresponding Weyl function $M(z)$ is defined by $M(z) := \Gamma'\gamma(z)$, i.e.

$$M(z) := a(z)(\Delta_d - \mathrm{d}\eta(z)), \qquad a(z) := \frac{1}{s(1;z)}. \tag{6}$$

There exists the following relation between the operators $H$ and $H^0$, see [5, section 1] for more details.

**Proposition 2.** *For $z \notin \operatorname{spec} H^0 \cup \operatorname{spec} H$ the operator $M(z) - A$ acting on $\mathcal{G}$ has a bounded inverse defined everywhere, and*

$$(H - z)^{-1} = (H^0 - z)^{-1} - \gamma(z)(M(z) - A)^{-1}\gamma(\bar z)^*. \tag{7}$$

*One has*

$$\operatorname{spec} H \setminus \operatorname{spec} H^0 = \{z \in \mathbb{R} \setminus \operatorname{spec} H^0 : 0 \in \operatorname{spec}(M(z) - A)\},$$

*and for $z \in \mathbb{R} \setminus \operatorname{spec} H^0$ there holds $\ker(H - z) = \gamma(z)\ker(M(z) - A)$. The maps $\gamma$ and $M$ depend analytically on their argument (outside of $\operatorname{spec} H^0$), $M(z)$ satisfies $M(\bar z) = M(z)^*$ and*

$$\textit{for any non-real } z \textit{ there is } c_z > 0 \qquad \textit{with} \quad \frac{\operatorname{Im} M(z)}{\operatorname{Im} z} \geqslant c_z \quad \textit{and} \tag{8}$$

$$M'(\lambda) = \gamma(\lambda)^*\gamma(\lambda) > 0 \qquad \text{for} \qquad \lambda \in \mathbb{R}\setminus \operatorname{spec} H^0. \tag{9}$$

*Furthermore,*

$$\gamma(z)^* f = 0 \qquad \textit{for any} \quad f \in \operatorname{ran}\gamma(z)^\perp. \tag{10}$$

We will also use a similar formula relating the resolvents of $H$ and $H_0$ (i.e. the operator corresponding to the zero coupling constants at all the vertices). Note that the operator $H_0$ formally corresponds to the case $d_1 = d$. As shown in [17], there holds $\operatorname{spec} H_0 = \operatorname{spec} L \cup \operatorname{spec} H_0$. Using theorem 1.32 in [5] one arrives at

**Proposition 3.** *Denote by $P$ the operator $\mathbb{Z}^d \ni (\mathbf{m}_1, \mathbf{m}_2) \mapsto \mathbf{m}_2 \in \mathbb{Z}^{d_2}$ and introduce the operators $\nu(z) = \gamma(z)P$ considered as a maps from $\mathbb{Z}^{d_2}$ to $\mathcal{H}$ as well as $B = -PA^{-1}P$ and $N(z) = -PM(z)^{-1}P$ considered as operators in $\mathbb{Z}^{d_2}$. Then there holds*

$$(H_0 - z)^{-1} - (H - z)^{-1} = \nu(z)(N(z) - B)^{-1}\nu^*(\bar z). \tag{11}$$

*The set* $\operatorname{spec} H \setminus \operatorname{spec} H_0$ *consists exactly of* $z \in \mathbb{R} \setminus \operatorname{spec} H_0$ *such that* $0 \in \operatorname{spec}(N(z) - B)$*, and the same correspondence holds for the eigenvalues, i.e.* $z \in \mathbb{R} \setminus \operatorname{spec} H^0$ *there holds* $\ker(H - z) = \nu(z) \ker(N(z) - B)$.

*For* $z \notin \operatorname{spec} H_0$*, the maps* $\nu(z)$ *and* $N(z)$ *satisfy the same properties listed in proposition 2 as* $\gamma(z)$ *and* $M(z)$.

By proposition 2, outside of the discrete set $\operatorname{spec} H^0$, the spectrum of $H$ consists of the real $z$ satisfying $0 \in \operatorname{spec}(M(z) - A)$ or, taking into account the explicit form (6),

$$0 \in \operatorname{spec}(a(z)(\Delta_d - d\eta(z)) - A).$$

Note that the operator on the right-hand side is exactly the Maryland-type model mentioned in section 1. In [2] the following was proved.

**Proposition 4.** *For any* $b \neq 0$ *the operator* $G := \Delta_d + \beta A$ *has the following spectral properties:*

*(a)* $[-2d, 2d] \subset \operatorname{spec} G$*;*
*(b) if the components of* $\boldsymbol{\omega}$ *are rationally independent, then the spectrum of G in the interval* $(-2d, 2d)$ *is purely absolutely continuous;*
*(c) for Diophantine* $\boldsymbol{\omega}$ *(see theorem 1) the rest of the real line is covered by the dense pure point spectrum.*

Transferring this proposition to the operator $M(z) - A$ one obtains

- $0 \in \operatorname{spec}(M(z) - A)$ for $|\eta(z)| \leqslant 2$;
- if the components of $\boldsymbol{\omega}$ are rationally independent, then the spectrum of $M(z) - A$ in the interval $a(z)(-2d - d\eta(z), 2d - d\eta(z))$ is purely absolutely continuous;
- for Diophantine $\boldsymbol{\omega}$ the rest of the real line is covered by the dense pure point spectrum.

Recall that the set $|\eta(z)| \leqslant 2$ coincides with the spectrum of $L$. Therefore, using the first of the above three properties and proposition 2 one immediately obtains

**Proposition 5.** $\operatorname{spec} L \subset \operatorname{spec} H$.

While for each fixed $z$ the operator $M(z)$ is of Maryland type, the dependence on $z$ is nonlinear. Furthermore, the proposition 2 does not allow to conclude about the spectral nature of $H$ from that of $M(z) - A$ (see [4] for a discussion of this correspondence within the framework of the general theory of self-adjoint extensions); as for now, such a relationship is only available under special conditions for the configuration and the coupling constants [5] which only hold in the case $d_1 = d$, i.e. for the zero coupling constants; the case $d_1 < d$ we are interested in is not covered. Hence to obtain a satisfactory description of the spectral properties of $H$ we need to repeat some constructions from [2, 9] and to combine them with the resolvent formulae given in propositions 2 and 3. For the study of the pure point spectrum for the full-space Maryland model on quantum graphs, this idea was used already in [18] and earlier in [10] for point interaction Hamiltonians in the Euclidean space.

## 4. The absolutely continuous spectrum

In this section we will first repeat some constructions for [2, 3] for the operator $M(z)$ taking into account the nonlinear dependence on the spectral parameter and then insert this into the resolvent formula of proposition 2.

Below we will use actively the Fourier transform. Denote $\mathbb{S}^1 := \{z \in \mathbb{C}, |z| = 1\}$ and $\mathbb{T}^n := \underbrace{\mathbb{S}^1 \times \ldots \times \mathbb{S}^1}_{n \text{ times}} \subset \mathbb{C}^n$. For $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \subset \mathbb{C}^n$ and $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n$ we write $\boldsymbol{\theta}^{\mathbf{p}} := \theta_1^{p_1} \ldots \theta_n^{p_n}$, and in this context $k \in \mathbb{Z}$ will be identified with the vector $(k, \ldots, k) \in \mathbb{Z}^n$, i.e. $\boldsymbol{\theta}^{-1} := \theta_1^{-1} \ldots \theta_l^{-1}$ etc. We denote by $F_n$ the Fourier transform carrying $\ell^2(\mathbb{Z}^n)$ to $\mathcal{L}^2(\mathbb{T}^n)$,

$$F_n \psi(\boldsymbol{\theta}) = \sum_{\mathbf{n} \in \mathbb{Z}^n} \psi(\mathbf{n}) \boldsymbol{\theta}^{\mathbf{n}}, \qquad F_n^{-1} f(\mathbf{n}) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} f(\boldsymbol{\theta}) \boldsymbol{\theta}^{-\mathbf{n}-1} \, \mathrm{d}\boldsymbol{\theta}.$$

Each $\boldsymbol{\theta} \in \mathbb{T}^d$ will be represented as $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ with $\boldsymbol{\theta}_1 \in \mathbb{T}^{d_1}$ and $\boldsymbol{\theta}_2 \in \mathbb{T}^{d_2}$.

Without loss of generality assume $g > 0$ (otherwise one can change the signs of $\boldsymbol{\omega}$ and $\varphi$). Consider the operator $L(z) := M(z) - A = M(z) + PvP$, where $v$ is an operator in $\ell^2(\mathbb{Z}^{d_2})$ acting as $vf(\mathbf{m}_2) = -g \tan(\boldsymbol{\omega}\mathbf{m}_2 + \varphi) f(\mathbf{m}_2)$, $\mathbf{m}_2 \in \mathbb{Z}^{d_2}$. For $\operatorname{Im} z \neq 0$ the operator $M(z)$ is invertible (as its imaginary part is non-degenerate) and one has

$$L(z)^{-1} = M(z)^{-1} - M(z)^{-1} T(z) M(z)^{-1}, \qquad T(z) = v - T(z) M(z)^{-1} v.$$

Obviously one can write $T(z) = Pt(z)P$ where the operator $t(z)$ acting in $\ell^2(\mathbb{Z}^{d_2})$ satisfies $t(z) = v + t(z)N(z)v$. Formally one has $t(z) = v(1 - N(z)v)^{-1}$ and it is needed to show that the operator in question is really invertible.

Let $U$ be the unitary operator in $\ell^2(\mathbb{Z}^{d_2})$ defined by the relation

$$(uf)(\mathbf{m}_2) = \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{\omega} \mathbf{m}_2} f(\mathbf{m}_2),$$

then, denoting $\chi := \mathrm{e}^{-2\pi \mathrm{i}\varphi}$, one can write

$$v = -\frac{g}{\mathrm{i}} \frac{1 - \chi U}{1 + \chi U}.$$

As $\operatorname{Im} N(z) \geqslant 0$ for $\operatorname{Im} z \geqslant 0$, the operator $\mathrm{i} + gN(z)$ is invertible for such $z$. Hence, for $\operatorname{Im} z \geqslant 0$ after a simple algebra one obtains

$$1 - N(z)v = (gN(z) + \mathrm{i})(1 - b(z)\chi U)(\mathrm{i}(1 + \chi U))^{-1}$$

where $b(z) = (gN(z) - \mathrm{i})(\mathrm{i} + gN(z))^{-1}$. In order to represent the inverse operator in terms of the Neumann series it is sufficient to show that $|b(z)| < 1$ for some $z$. To see this, it is useful to pass to the Fourier representation.

For $\lambda \in \mathbb{C}$ denote $G_d(\lambda) := (\Delta_d - \lambda)^{-1}$. Recall that in the Fourier representation $\Delta_d$ becomes the multiplication by the function $\Delta_d(\boldsymbol{\theta}) = \sum_j \left(\theta_j + \theta_j^{-1}\right)$, hence the matrix of $G_d(\lambda)$ is given by

$$G_d(\mathbf{m} - \mathbf{m}'; \lambda) = \frac{1}{(2\pi \mathrm{i})^d} \int_{\mathbb{T}^d} \frac{\boldsymbol{\theta}^{-(\mathbf{m}-\mathbf{m}')-1} \, \mathrm{d}\boldsymbol{\theta}}{\Delta_d(\boldsymbol{\theta}) - \lambda}.$$

On the other hand, the matrix of the operator $N(z)$ is $N(\mathbf{m}_2 - \mathbf{m}_2'; z) = -a(z)^{-1} G_d((0, \mathbf{m}_2) - (0, \mathbf{m}_2'); \mathrm{d}\eta(z))$, hence

$$\begin{aligned} N(\mathbf{m}_2 - \mathbf{m}_2'; z) &= -a(z)^{-1} \frac{1}{(2\pi \mathrm{i})^d} \int_{\mathbb{T}^{d_2}} \boldsymbol{\theta}_2^{-(\mathbf{m}_2 - \mathbf{m}_2')-1} \, \mathrm{d}\boldsymbol{\theta}_2 \int_{\mathbb{T}^{d_1}} \frac{\boldsymbol{\theta}_1^{-1} \, \mathrm{d}\boldsymbol{\theta}_1}{\Delta_d(\boldsymbol{\theta}) - \mathrm{d}\eta(z)} \\ &= -a(z)^{-1} \frac{1}{(2\pi \mathrm{i})^{d_2}} \int_{\mathbb{T}^{d_2}} G_{d_1}(\mathbf{0}; \mathrm{d}\eta(z) - \Delta_{d_2}(\boldsymbol{\theta}_2)) \boldsymbol{\theta}_2^{-(\mathbf{m}_2 - \mathbf{m}_2')-1} \, \mathrm{d}\boldsymbol{\theta}_2. \end{aligned} \tag{12}$$

In particular, it is clear that in the Fourier representation $N(z)$ is the multiplication by the function

$$N(\boldsymbol{\theta}_2; z) = -a(z)^{-1} G_{d_1}(\mathbf{0}; \mathrm{d}\eta(z) - \Delta_{d_2}(\boldsymbol{\theta}_2)). \tag{13}$$

As $\operatorname{Im} N(z) > 0$ for $\operatorname{Im} z > 0$, the imaginary part $\operatorname{Im} N(\boldsymbol{\theta}_2; z)$ is positive for such $z$. The operator $b(z)$ in the Fourier representation becomes the multiplication by the function

$$b(\boldsymbol{\theta}_2, z) = \frac{gN(\boldsymbol{\theta}_2, z) - \mathrm{i}}{gN(\boldsymbol{\theta}_2, z) + \mathrm{i}},$$

hence $\|b(z)\| \equiv \sup_{\boldsymbol{\theta}_2} |b(\boldsymbol{\theta}_2, z)| < 1$ for $\operatorname{Im} z > 0$. Therefore, one can represent

$$\begin{aligned} t(z) &= v(1 - N(z)v)^{-1} \\ &= -g(1 - \chi U)(1 - b(z)\chi U)^{-1}(gN(z) + \mathrm{i})^{-1} \\ &= -g(1 - \chi U)\sum_{m=0}^{\infty} \chi^m (b(z)U)^m \\ &= -g(gN(z) + \mathrm{i})^{-1}\left(1 - 2\mathrm{i}\sum_{m=1}^{\infty}(gN(z) + \mathrm{i})^{-1}U(b(z)U)^{m-1}\right) \end{aligned} \tag{14}$$

and one has

$$(M(z) - A)^{-1} = M(z)^{-1} - M(z)^{-1}Pt(z)PM(z)^{-1}.$$

After these preparations we can prove

**Proposition 6.** *Denote $I := \eta^{-1}((-2, 2))$. If the vector $\boldsymbol{\omega}$ has rationally independent components, then the operator $H$ has only absolutely continuous spectrum in $I$.*

**Proof.** According to the general spectral theory we need to show that there exists a dense subset $\mathcal{L}$ of $\mathcal{H}$ such that the limit $\lim_{\varepsilon\to 0+} \operatorname{Im}\langle f, (H - \lambda - \mathrm{i}\varepsilon)^{-1} f\rangle$ exists and is finite for all $g \in \mathcal{L}$ and $\lambda \in I$.

Represent

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1, \qquad \mathcal{H}_0 := \left(\bigcup_{\operatorname{Im} z\neq 0} \gamma(z)(\ell^2(\mathbb{Z}^d))\right)^{\perp}, \qquad \mathcal{H}_1 := \mathcal{H}_0^{\perp}; \tag{15}$$

in other words, $\mathcal{H}_1$ is the closure of the linear hull of the set $\{\gamma(z)\varphi : \operatorname{Im} z \neq 0, \varphi \in \ell^2(\mathbb{Z}^d)\}$.

By the Krein resolvent formula, for any $f \in \mathcal{H}_0$ and any $z$ with $\operatorname{Im} z \neq 0$ one has $\gamma^*(z)f = 0$. Hence, by (7), there holds $(H - z)^{-1}f = (H^{0-z})^{-1}f$, hence $\lim_{\varepsilon\to 0+} \operatorname{Im}\langle f, (H - \lambda - i\varepsilon)^{-1} f\rangle = 0$, because $(H^0 - \lambda)^{-1}$ is a bounded self-adjoint operator.

Consider the vectors $f = \gamma(\zeta)h$ for $h = (M(\zeta) - A)^{-1}\xi$, $\operatorname{Im}\zeta \neq 0$. These vectors form a dense subset in $\mathcal{H}_1$ as $\xi$ runs over a dense subset of $\ell^2(\mathbb{Z}^d)$. By elementary calculations (see e.g. section 3 in [5]) one can write

$$(H - \lambda - \mathrm{i}\varepsilon)^{-1} f = \frac{1}{\zeta - \lambda - \mathrm{i}\varepsilon}(f - \gamma(\lambda + \mathrm{i}\varepsilon)(M(\lambda + \mathrm{i}\varepsilon) - A)^{-1}\xi).$$

Hence it is sufficient to show that $\lim_{\varepsilon\to 0+} \operatorname{Im}\langle\delta_{\mathbf{m}}, (M(\lambda + \mathrm{i}\varepsilon) - A)^{-1}\delta_{\mathbf{m}}\rangle$ exists and is finite for any $\mathbf{m} \in \mathbb{Z}^d$. In view of the series representation for $(M(z) - A)^{-1}$ it is sufficient to show that the series converges for real $z \in I$ and not only for $\operatorname{Im} z > 0$. On the other hand, $(M(z) - A)^{-1} = a(z)^{-1}(\Delta_d - \mathrm{d}\eta(z) - a(z)^{-1}A)^{-1}$, and it is shown in [2, theorem 3.1] that $\operatorname{Im}\langle\delta_{\mathbf{m}}, (\Delta_d - c - bA)^{-1}\delta_{\mathbf{m}}\rangle$ exists and is finite for any $c \in (-2d, 2d)$ and any $b \in \mathbb{R}$. This completes the proof. $\square$

## 5. The pure point spectrum

In this section we will use the second version of the resolvent formula, equation (11). Hence for $z \notin \operatorname{spec} H_0$ we have the equivalence $z \in \operatorname{spec} H$ iff $0 \in \operatorname{spec}(N(z) - B)$. Here $N$ is a translationally invariant operator in $\ell^2(\mathbb{Z}^{d_2})$ whose matrix elements are given by (12), and the operator $B$, as already mentioned below, in the multiplication by the sequence $g^{-1} \tan \pi(\boldsymbol{\omega}\mathbf{m}_2 + \varphi + 1/2)$. It is useful to set $g' = -g, \boldsymbol{\omega}' := -\boldsymbol{\omega}, \varphi' := -\varphi - 1/2$, then $B$ becomes a multiplication by $-g' \tan \pi(\boldsymbol{\omega}'\mathbf{m}_2 + \varphi')$ with $g' > 0$.

To alleviate the notation, below we will write $d$ instead of $d_2$ and drop the indices for $g', \boldsymbol{\omega}'$ and $\varphi'$ as this does not lead to confusions.

In [9] the operators of the following form were considered: $G = G_0 + B$, where $G_0$ is a translationally invariant operator in $\ell^2(\mathbb{Z}^d)$ given by

$$G\xi(\mathbf{m}) = \sum_{\mathbf{m}' \in \mathbb{Z}^d} a(\mathbf{m} - \mathbf{m}')\xi(\mathbf{m}'),$$

with the coefficients satisfying an exponential bound

$$|a(\mathbf{m})| \leqslant c_1 \exp(-c_2|\mathbf{m}|), \qquad c_1, c_2 > 0, \qquad \text{for all} \quad \mathbf{m} \in \mathbb{Z}^d.$$

It was shown that the spectrum of such operators covers the whole real line and is pure point. For each fixed $z \notin \operatorname{spec} H_0$ the reduced operator $N(z) - B$ is of the above type and hence has a pure point spectrum dense everywhere. On the other hand, one is not able to transfer these results directly to the quantum graph Hamiltonian $H$ using just the machinery of self-adjoint extensions. Hence, like in the previous section, we need to repeat some constructions of [9] and then combine them with the resolvent formula of proposition 3. Note that the constructions below appear to be almost identical to those of [18] where we studied the full space quantum graph Maryland model (but with a different operator $N$) because of the similar structure of the resolvent.

Introduce the operators

$$D(z) := (N(z) - \mathrm{i}g)^{-1}, \qquad C(z) := -(N(z) + \mathrm{i}g)(N(z) - \mathrm{i}g)^{-1},$$

they are defined at least for $z$ with $\operatorname{Re} z \notin \operatorname{spec} H_0$ and $|\operatorname{Im} z|$ sufficiently small. One can write for such $z$ the identity

$$N(z) - B = D(z)^{-1}(1 - \chi C(z)U)(1 + \chi U)^{-1}. \tag{16}$$

Recall that under the Fourier transform $N(z)$ becomes the multiplication by the function $N(\boldsymbol{\theta}, z)$ given by (13), the operators $D(z)$ and $C(z)$ become the multiplications by $D(\boldsymbol{\theta}, z) := (N(\boldsymbol{\theta}, z) - \mathrm{i}g)^{-1}$ by $C(\boldsymbol{\theta}, z) := -(N(\boldsymbol{\theta}, z) + \mathrm{i}g)(N(\boldsymbol{\theta}, z) - \mathrm{i}g)^{-1}$, respectively, and $U$ becomes a shift operator, $Uk(\boldsymbol{\theta}) = k(\mathrm{e}^{-2\pi \mathrm{i}\omega_1}\theta_1, \ldots, \mathrm{e}^{-2\pi \mathrm{i}\omega_d}\theta_d)$.

Consider an arbitrary segment $[a, b] \subset \mathbb{R} \setminus \operatorname{spec} H_0$. Recall that the spectrum of $H_0$ coincides with the spectrum of $L$ up to the discrete set $\operatorname{spec} H^0$. Equation (8), the analyticity of $\gamma$, and the self-adjointness of $N(z)$ for real $z$ imply the existence of $\delta' > 0$ such that $\|\operatorname{Im} N(z)\| \leqslant g/2$ for $z \in Z := \{z \in \mathbb{C} : |\operatorname{Im} z| \leqslant \delta', \operatorname{Re} z \in [a, b]\}$. At the same time, this means that $|\operatorname{Im} N(\boldsymbol{\theta}, z)| \leqslant g/2$ for $z \in Z$. As follows from the integral representation, $N(\boldsymbol{\theta}, z)$ can be continued to an analytic function in $Z \times \Theta, \Theta := \{\boldsymbol{\theta} \subset \mathbb{C}^d : r < |\theta_j| < R\}, 0 < r < 1 < R < \infty$. Choosing $r$ and $R$ sufficiently close to 1 one immediately sees that the function

$$C(\boldsymbol{\theta}, z) := \frac{g^2 - (\operatorname{Im} N(\boldsymbol{\theta}, z))^2 - (\operatorname{Re} N(\boldsymbol{\theta}, z))^2 - 2\mathrm{i}g \operatorname{Re} N(\boldsymbol{\theta}, z)}{|N(\boldsymbol{\theta}, z) - \mathrm{i}g|^2}$$

does not take values in $(-\infty, 0)$ for $(\boldsymbol{\theta}, z) \in \Theta \times Z$. Therefore, the function $f(\boldsymbol{\theta}, z) := \log C(\boldsymbol{\theta}, z)$ is well defined and analytic in $\Theta \times Z$, where log denotes the principal branch of the logarithm. The Diophantine property (5) implies (see [9, lemma 3.2]) that the operator $1 - U$ is a bijection on the set of functions $v$ analytic in $\Theta$ with

$$\int_{\mathbb{T}^d} v(\boldsymbol{\theta})\boldsymbol{\theta}^{-1}\,\mathrm{d}\boldsymbol{\theta} = 0.$$

Hence the function $t(\boldsymbol{\theta}, z) := (1-U)^{-1}(f(\boldsymbol{\theta}, z) - f_0(z))$ is well defined and analytic in $Z \times \Theta$, where

$$f_0(z) := \frac{1}{(2\pi\mathrm{i})^d}\int_{\mathbb{T}^d} f(\boldsymbol{\theta}, z)\boldsymbol{\theta}^{-1}\,\mathrm{d}\boldsymbol{\theta}. \tag{17}$$

**Lemma 7.** *The function $f_0$ is analytic in $Z$,*

$$\operatorname{Re} f_0(z) < 0 \qquad \text{for} \quad \operatorname{Im} z > 0, \tag{18}$$

$$\operatorname{Re} f(\boldsymbol{\theta}, z) = \operatorname{Re} t(\boldsymbol{\theta}, z) = \operatorname{Re} f_0(z) = 0 \qquad \text{for} \quad \operatorname{Im} z = 0. \tag{19}$$

*For real $\lambda$ one has $f_0(\lambda) = 2\mathrm{i}\sigma(\lambda)$, where*

$$\sigma(\lambda) = \frac{1}{(2\pi\mathrm{i})^d}\int_{\mathbb{T}^d} \arctan\frac{N(\boldsymbol{\theta}, \lambda)}{g}\boldsymbol{\theta}^{-1}\,\mathrm{d}\boldsymbol{\theta}.$$

*The function $\sigma$ is real valued, strictly increasing, and continuously differentiable on $[a, b]$.*

**Proof.** The analyticity of $f_0$ follows from its integral representation. Equation (18) follows from (17) if one takes into account the inequalities $\operatorname{Im} N(\boldsymbol{\theta}, z) > 0$ for $\operatorname{Im} z > 0$ and $\operatorname{Re}\log z < 0$ for $|z| < 1$. Equalities (18) follows from (17) and the real-valuedness of $N(\boldsymbol{\theta}, z)$ for real $z$.

By elementary calculations, for $x \in \mathbb{R}$ and $y > 0$ one has

$$g_1(x) := \frac{1}{2\mathrm{i}}\log\frac{\mathrm{i}y + x}{\mathrm{i}y - x} \equiv \arctan\frac{x}{y} =: g_2(x). \tag{20}$$

In fact, this follows from

$$g_1'(x) = g_2'(x) = \frac{y}{x^2 + y^2} \tag{21}$$

and $g_1(0) = g_2(0) = 0$. Equation (20) obviously implies $f_0(\lambda) = 2\mathrm{i}\sigma(\lambda)$ for $\lambda \in \mathbb{R}$. Furthermore, as follows from (21),

$$\sigma'(\lambda) = \frac{1}{(2\pi\mathrm{i})^d}\int_{\mathbb{T}^d} \frac{gN_\lambda'(\boldsymbol{\theta}, \lambda)}{N(\boldsymbol{\theta}, \lambda)^2 + g^2}\boldsymbol{\theta}^{-1}\,\mathrm{d}\boldsymbol{\theta},$$

and, by (9), $\sigma'(\lambda) > 0$. □

An immediate corollary of the analyticity of $f_0$ and of (18) is

**Lemma 8.** *There exists $\varepsilon_0 > 0$ such that $|\mathrm{e}^{f_0(\lambda)}\xi - 1| \leqslant 2|\mathrm{e}^{f_0(\lambda+\mathrm{i}\varepsilon)}\xi - 1|$ for all $\xi \in \mathbb{S}^1$, $\lambda \in [a, b]$, and $\varepsilon \in [0, \varepsilon_0]$.*

Denote by $t(z)$ and $f(z)$ the multiplication operators by $t(\boldsymbol{\theta}, z)$ and $f(\boldsymbol{\theta}, z)$ in $\mathcal{L}^2(\mathbb{T}^d)$, respectively. By definition of $t(\boldsymbol{\theta}, z)$ for any $xi \in \mathcal{L}^2(\mathbb{T}^d)$

$$\begin{aligned}
\mathrm{e}^{t(z)}\,\mathrm{e}^{f_0(z)}U\,\mathrm{e}^{-t(z)}\xi(\boldsymbol{\theta}) &= \mathrm{e}^{t(\boldsymbol{\theta},z)}\mathrm{e}^{f_0(\boldsymbol{\theta},z)}\exp(-t(z, \mathrm{e}^{-2\pi\mathrm{i}\omega_1}\theta_1, \ldots, \mathrm{e}^{-2\pi\mathrm{i}\omega_d}\theta_d))U\varphi(\boldsymbol{\theta})\\
&= \exp(t(\boldsymbol{\theta}, z) - Ut(\boldsymbol{\theta}, z) + f_0(\boldsymbol{\theta}, z))U\varphi(\boldsymbol{\theta}, z)\\
&= \mathrm{e}^{f(z)}U\varphi(\boldsymbol{\theta}) = C(z)U\varphi(\boldsymbol{\theta}).
\end{aligned} \tag{22}$$

Therefore, one can rewrite equation (16) as

$$N(z) - B = D(z)^{-1}\,\mathrm{e}^{t(z)}(1 - \mathrm{e}^{f_0(z)}\chi U)\,\mathrm{e}^{-t(z)}(1 + \chi U)^{-1}. \tag{23}$$

**Proposition 9.** *The set of the eigenvalues of $H$ in $[a, b]$ is dense and coincides with the set of solutions $\lambda$ to*

$$\sigma(\lambda) = \pi(\boldsymbol{\omega}\mathbf{m} + \varphi) \bmod \pi, \qquad \mathbf{m} \in \mathbb{Z}^d. \tag{24}$$

*Each of these eigenvalues is simple, and for any fixed $\mathbf{m} \in \mathbb{Z}^d$ equation (24) has at most one solution $\lambda(\mathbf{m})$, and $\lambda(\mathbf{m}) \neq \lambda(\mathbf{m}')$ for $\mathbf{m} \neq \mathbf{m}'$.*

**Proof.** As follows from proposition 2 and the resolvent formula (11), the eigenvalues $\lambda$ of $H$ outside $\operatorname{spec} H_0$ are determined by the condition $\ker(N(\lambda) - B) \neq 0$, and their multiplicity coincides with the dimension of the corresponding kernels. Equation (22) shows that the condition $(N(\lambda) - B)u = 0$ is equivalent to $(1 - \mathrm{e}^{f_0(\lambda)}\chi U)\,\mathrm{e}^{-t(\lambda)}(1 + \chi U)^{-1}u = 0$ or, denoting $v := \mathrm{e}^{-t(\lambda)}(1 + \chi U)^{-1}u$, $(1 - \mathrm{e}^{f_0(\lambda)}\chi U)v = 0$, which can be rewritten as

$$\chi U v = \mathrm{e}^{-f_0(\lambda)}v, \qquad v \neq 0. \tag{25}$$

As $\chi U$ has the simple eigenvalues $\mathrm{e}^{-2\pi\mathrm{i}(\boldsymbol{\omega}\mathbf{m}+\varphi)}$, $\mathbf{m} \in \mathbb{Z}^d$, and the corresponding eigenvectors form a basis, equation (25) implies (24) if one takes into account the identity $f_0(\lambda) = 2\mathrm{i}\sigma(\lambda)$ proved in lemma 7. The rest follows from the monotonicity of $\sigma$, the inclusion $\operatorname{ran}\sigma \subset (-\pi/2, \pi/2)$, and the arithmetic properties of $\boldsymbol{\omega}$ and $\varphi$, see (1) and (5). □

As $[a, b]$ was an arbitrary interval from $\mathbb{R} \setminus \operatorname{spec} H_0$, one has an immediate corollary.

**Proposition 10.** *The pure point spectrum of $H$ is dense in $\mathbb{R} \setminus \operatorname{spec} H_0$.*

Now it remains to show that the spectrum of $H$ in the interval considered is pure point.

Take some $\alpha > 0$. For any $\delta > 0$ we denote

$$\mathbb{S}^1_\delta = \bigcup_{m\in\mathbb{Z}^d} \{\xi \in \mathbb{S}^1 : |\operatorname{Arg}\xi - \operatorname{Arg}\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{\omega}\mathbf{m}}| \leqslant \delta(1 + |\mathbf{m}|)^{-d-\alpha}\}, \qquad \widetilde{\mathbb{S}}^1_\delta := \mathbb{S}^1 \setminus \mathbb{S}^1_\delta.$$

Clearly, there holds

$$|1 - \xi\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{\omega}\mathbf{m}}| \geqslant 2\pi^{-1}\delta(1 + |\mathbf{m}|)^{-d-\alpha}, \qquad \xi \in \widetilde{\mathbb{S}}^1_\delta, \qquad m \in \mathbb{Z}^d. \tag{26}$$

Let $\Delta \subset [a, b]$ be an interval whose ends are not eigenvalues of $H$. Consider the mapping $h : \lambda \mapsto \chi\,\mathrm{e}^{f_0(\lambda)}$. By lemma 7, $h$ is a diffeomorphism between $\Delta$ and $h(\Delta)$. By proposition 9 one has $h(\lambda(\mathbf{m})) = \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{\omega}\mathbf{m}}$. Take an arbitrary $\delta > 0$ and denote

$$\Delta_\delta := \Delta \cap h^{-1}(\mathbb{S}^1_\delta), \qquad \widetilde{\Delta}_\delta := \Delta \cap h^{-1}(\widetilde{\mathbb{S}}^1_\delta) \equiv \Delta\backslash\Delta_\delta.$$

Clearly, $\Delta_\delta$ is a countable union of intervals and the limit set $\bigcap_{\delta>0}\Delta_\delta$ coincides with the set of all the eigenvalues $\bigcup_m\{\lambda(m)\}$.

**Lemma 11.** *There exists $\varepsilon_0 > 0$ such that for any $\delta > 0$ and any $\mathbf{n} \in \mathbb{Z}^d$ there exists $C > 0$ such that*

$$\|(N(\lambda + \mathrm{i}\varepsilon) - B)^{-1}\delta_{\mathbf{n}}\| \leqslant C \tag{27}$$

*for all $\lambda \in \widetilde{\Delta}_\delta$, and $\varepsilon \in (0, \varepsilon_0)$.*

**Proof.** Rewrite equation (23) in the form

$$(N(z) - B)^{-1} = (1 + \chi U)\,\mathrm{e}^{t(z)}(1 - \mathrm{e}^{f_0(z)}\chi U)^{-1}\,\mathrm{e}^{-t(z)}D(z).$$

Note that the Fourier transform of $\delta_{\mathbf{n}}$ is the function $\boldsymbol{\theta} \mapsto \boldsymbol{\theta}^{\mathbf{n}}$. Denote $\Psi(z;\boldsymbol{\theta}) := \mathrm{e}^{-t(\boldsymbol{\theta},z)}B(\boldsymbol{\theta},z)\boldsymbol{\theta}^{\mathbf{n}}$. Due to the analyticity one can estimate uniformly in $Z$:

$$|\psi_z(\mathbf{m})| \leqslant C'\,\mathrm{e}^{-\rho|\mathbf{m}|}, \qquad C',\rho>0, \qquad \psi_z := F_d^{-1}\Psi, \qquad \|(1+\chi U)\,\mathrm{e}^{t(z)}\| \leqslant C'.$$

Therefore, (27) follows from the inequality

$$\|(1-\mathrm{e}^{f_0(\lambda+\mathrm{i}\varepsilon)}\chi U)^{-1}\Psi\| \leqslant C. \tag{28}$$

Assume that $\varepsilon_0$ satisfies the conditions of lemma 8, then uniformly for $\lambda\in\Delta$ and $\varepsilon\in(0,\varepsilon_0)$ one has

$$\begin{aligned}\left|\left(F_d^{-1}(1-\mathrm{e}^{f_0(\lambda+\mathrm{i}\varepsilon)}\chi U)^{-1}\Psi\right)(\mathbf{m})\right| &= |(1-\mathrm{e}^{f_0(\lambda+\mathrm{i}\varepsilon)}\chi\,\mathrm{e}^{2\pi\mathrm{i}\omega\mathbf{m}})^{-1}\psi_{\lambda+\mathrm{i}\varepsilon}(\mathbf{m})|\\ &\leqslant 2|(1-\mathrm{e}^{f_0(\lambda)}\chi\,\mathrm{e}^{2\pi\mathrm{i}\omega\mathbf{m}})^{-1}|\cdot|\psi_{\lambda+\mathrm{i}\varepsilon}(\mathbf{m})|.\end{aligned}$$

As in our case $h(\lambda)\equiv\chi\,\mathrm{e}^{f_0(\lambda)}\in\widetilde{\mathbb{S}}^1_\delta$, due to (26) we have

$$|(1-\mathrm{e}^{f_0(\lambda)}\chi\,\mathrm{e}^{-2\pi\mathrm{i}\omega\mathbf{m}})^{-1}| \leqslant \frac{\pi}{2\delta}(1+|\mathbf{m}|)^{d+\alpha}.$$

Finally,

$$\begin{aligned}\|(1-\mathrm{e}^{f_0(\lambda+i\varepsilon)}\chi U)^{-1}\Psi\|^2 &= \sum_{\mathbf{m}\in\mathbb{Z}^d}\left|\left(F_d^{-1}(1-\mathrm{e}^{f_0(\lambda+\mathrm{i}\varepsilon)}\chi U)^{-1}\Psi\right)(\mathbf{m})\right|^2\\ &\leqslant\left(\frac{\pi C'}{\delta}\right)^2\sum_{\mathbf{m}\in\mathbb{Z}^d}(1+|\mathbf{m}|)^{2(d+\alpha)}\,\mathrm{e}^{-2\rho|\mathbf{m}|}<\infty,\end{aligned}$$

and (28) is proved. □

Now we are able to estimate the spectral projections corresponding to $H$.

**Lemma 12.** *For any $f\in\mathcal{H}$ and any $\delta>0$ one has*

$$\lim_{\varepsilon\to0+}\varepsilon\int_{\widetilde{\Delta}_\delta}\|(H-\lambda-\mathrm{i}\varepsilon)^{-1}f\|^2\,\mathrm{d}\lambda=0. \tag{29}$$

**Proof.** Here we are going to use proposition 2. First note that due to $\widetilde{\Delta}_\delta\subset\mathbb{R}\setminus\operatorname{spec}H_0$ one has

$$\lim_{\varepsilon\to0}\varepsilon\int_{\widetilde{\Delta}_\delta}\|(H_0-\lambda-\mathrm{i}\varepsilon)^{-1}f\|^2\mathrm{d}\lambda=0 \qquad \text{for any}\quad f\in\mathcal{H}. \tag{30}$$

Similar to (15) let us consider the decomposition

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1, \qquad \mathcal{H}_0:=\left(\bigcup_{\operatorname{Im}z\neq0}\nu(z)(\ell^2(\mathbb{Z}^d))\right)^{\perp}, \qquad \mathcal{H}_1:=\mathcal{H}_0^{\perp};$$

As previously, by (11), for any $f\in\mathcal{H}_0$ and any $z$ with $\operatorname{Im}z\neq0$ one has $\nu^*(z)f=0$. Hence, by (7), there holds $(H-z)^{-1}f=(H_0-z)^{-1}$ and (30) implies (29) for $f\in\mathcal{H}_0$.

Now it is sufficient to show (30) for vectors $f=\nu(\zeta)h$ for $h=(N(\zeta)-B)^{-1}\delta_{\mathbf{m}}$, $\mathbf{m}\in\mathbb{Z}^d$, $\operatorname{Im}\zeta\neq0$. The operators $(N(\zeta)-B)^{-1}$ have dense range (coinciding with $\operatorname{dom}B$), hence the linear hull of such vectors $f$ is dense in $\mathcal{H}_1$. By elementary calculations (see e.g. section 3 in [5]) one rewrites equation (7) as

$$(H-\lambda-\mathrm{i}\varepsilon)^{-1}f=\frac{1}{\zeta-\lambda-\mathrm{i}\varepsilon}(f-\nu(\lambda+\mathrm{i}\varepsilon)(N(\lambda+\mathrm{i}\varepsilon)-B)^{-1}\delta_{\mathbf{m}}). \tag{31}$$

Due to lemma 11 we have $\|(N(\lambda+\mathrm{i}\varepsilon)-B)^{-1}\delta_{\mathbf{m}}\| \leqslant C$ with some $C>0$, for all $\lambda\in\widetilde{\Delta}_\delta$ and sufficiently small $\varepsilon$, and (31) implies

$$\|(H-\lambda-\mathrm{i}\varepsilon)^{-1}f\| \leqslant \frac{\|f\|+C\|\nu(\lambda+\mathrm{i}\varepsilon)\|}{|\zeta-\lambda-\mathrm{i}\varepsilon|},$$

and due to the analyticity of $\gamma$, one can estimate $\|(H-\lambda-\mathrm{i}\varepsilon)^{-1}f\|\leqslant C'$ with some $C'>0$ for all $\lambda\in\widetilde{\Delta}_\delta$ and sufficiently small $\varepsilon$. This obviously implies (29). □

**Proposition 13.** *The spectrum of H outside* spec $L$ *is pure point.*

**Proof.** We are going to show that for any $f\in\mathcal{H}$ and any interval $\Delta\subset\mathbb{R}\setminus\operatorname{spec}H_0$ the spectral measure $\mu_f$ associated with $H$ and $f$ satisfies $\mu_f(\Delta)=\mu_f\big(\Delta\cap\bigcup_m\{\lambda(m)\}\big)$; this proves that all the spectral measures are pure point.

By the Stone formula, for any set $X$ which is a countable union of intervals whose ends are not eigenvalues of $H$ one has

$$\mu_f(X)=\lim_{\varepsilon\to 0+}\frac{\varepsilon}{\pi}\int_X\|(H-\lambda-\mathrm{i}\varepsilon)f\|^2\mathrm{d}\lambda.$$

Using lemma 12, for any $\delta>0$ we estimate

$$\begin{aligned}\mu_f(\Delta)&=\lim_{\varepsilon\to 0+}\frac{\varepsilon}{\pi}\int_\Delta\|(H-\lambda-\mathrm{i}\varepsilon)f\|^2\,\mathrm{d}\lambda\\&=\lim_{\varepsilon\to 0+}\frac{\varepsilon}{\pi}\int_{\Delta_\delta}\|(H-\lambda-\mathrm{i}\varepsilon)f\|^2\,\mathrm{d}\lambda+\lim_{\varepsilon\to 0+}\frac{\varepsilon}{\pi}\int_{\widetilde{\Delta}_\delta}\|(H-\lambda-\mathrm{i}\varepsilon)f\|^2\,\mathrm{d}\lambda\\&=\lim_{\varepsilon\to 0+}\frac{\varepsilon}{\pi}\int_{\Delta_\delta}\|(H-\lambda-\mathrm{i}\varepsilon)f\|^2\,\mathrm{d}\lambda=\mu_f(\Delta_\delta).\end{aligned}$$

As $\delta$ is arbitrary and $\bigcap_{\delta>0}\Delta_\delta=\bigcup_m\{\lambda(m)\}$, the theorem is proved. □

## Acknowledgments

The work was supported by the Marie Curie Intra-European Fellowship PIEF-GA-2008-219641 during my stay at the University Paris Nord in July–September 2008.

## References

[1] Bellissard J, Lima R and Scoppola E 1983 Localization in $\nu$-dimensional incommensurate structures *Commun. Math. Phys.* **88** 465–77

[2] Bentosela F, Briet Ph and Pastur L 2003 On the spectral and wave propagation properties of the surface Maryland model *J. Math. Phys.* **44** 1–35

[3] Bentosela F, Briet Ph and Pastur L 2005 Spectral analysis of the generalized surface Maryland model *St. Petersbg. Math. J.* **16** 923–42

[4] Brasche J F, Malamud M M and Neidhardt H 2002 Weyl function and spectral properties of self-adjoint extensions *Integr. Eqns Operator Theory* **43** 264–89

[5] Brüning J, Geyler V and Pankrashkin K 2008 Spectra of self-adjoint extensions and applications to solvable Schrödinger operators *Rev. Math. Phys.* **20** 1–70

[6] Exner P 1995 Lattice Kronig–Penney models *Phys. Rev. Lett.* **74** 3503–6

[7] Exner P and Fraas M 2007 On the dense point and absolutely continuous spectrum for Hamiltonians with concentric $\delta$ shells *Lett. Math. Phys.* **82** 25–37

[8] Exner P, Hejčík P and Šeba P 2006 Approximations by graphs and emergence of global structures *Rep. Math. Phys.* **57** 445–55

[9] Figotin A L and Pastur L A 1984 An exactly solvable model of a multidimensional incommensurate structure *Commun. Math. Phys.* **95** 401–25

[10] Geyler V A and Margulis V A 1987 Anderson localization in the nondiscrete Maryland model *Theor. Math. Phys.* **70** 133–40

[11] Gnutzmann S and Smilansky U 2006 Quantum graphs: applications to quantum chaos and universal spectral statistics *Adv. Phys.* **55** 527–625

[12] Grempel D R, Fishman S and Prange R E 1982 Localization in an incommensurate potential: an exactly solvable model *Phys. Rev. Lett.* **49** 833–6

[13] Khoruzhenko B A and Pastur L A 1997 The localization of surface states: an exactly solvable model *Phys. Rep.* **288** 109–26

[14] Kuchment P 2004 Quantum graphs: I. Some basic structures *Waves Random Media* **14** S107–28

[15] Kuchment P 2005 Quantum graphs: II. Some spectral properties of quantum and combinatorial graphs *J. Phys. A: Math. Gen.* **38** 4887–900

[16] Melnikov Yu and Pavlov B 2001 Scattering on graphs and one-dimensional approximations to *N*-dimensional Schrödinger operators *J. Math. Phys.* **42** 1202–28

[17] Pankrashkin K 2006 Spectra of Schrödinger operators on equilateral quantum graphs *Lett. Math. Phys.* **77** 139–54

[18] Pankrashkin K 2008 Localization in a quasiperiodic model on quantum graphs *Analysis on Graphs and its Applications* (*Proc. Symp. Pure Math.* vol 77) ed P Exner *et al* (Providence, RI: American Mathematical Society) pp 459–67